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# Moutard transform approach to generalized analytic functions with contour poles <sup>\*</sup>

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## Abstract

We continue studies of Moutard-type transforms for the generalized analytic functions started in [13], [14]. In particular, we show that generalized analytic functions with the simplest contour poles can be Moutard transformed to the regular ones, at least, locally. In addition, the later Moutard-type transforms are locally invertible.

## 1 Introduction

We study the equations

$$\partial_{\bar{z}}\psi = u\bar{\psi} \quad \text{in } D, \quad (1)$$

$$\partial_{\bar{z}}\psi^+ = -\bar{u}\bar{\psi}^+ \quad \text{in } D, \quad (2)$$

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where  $D$  is an open domain in  $\mathbb{C}$ ,  $u = u(z)$  is a given function in  $D$ ,  $\partial_{\bar{z}} = \partial/\partial\bar{z}$ . The functions  $\psi = \psi(z)$  satisfying equation (1) are known as generalized analytic functions in  $D$ , equation (2) is known as the conjugate equation to (1); see [30]. In the present article the notation  $f = f(z)$  does not mean that  $f$  is holomorphic.

The classical theory of generalized analytic functions is presented in [3], [30]. In addition, very recently in [13], [14] it was shown that a new progress in this theory is possible by involving ideas of Moutard-type transforms going back to [22]. Actually, ideas of Moutard-type transform were developed and successfully used in the soliton theory in dimension 2+1, in the spectral theory in dimension 2 and in the differential geometry; see [5], [21], [23], [26]–[29] and references therein.

We recall that in our case the Moutard-type transforms assign in quadratures to a given coefficient  $u$  and fixed solutions  $f_j, f_j^+, j = 1, \dots, N$ , of equations (1),(2), and all formal solutions  $\psi, \psi^+$  of (1), (2) a new coefficient  $\tilde{u}$  and new related formal solutions  $\tilde{\psi}, \tilde{\psi}^+$  for these generalized analytic function equations; see [13], [14]. In turn, the construction of [13] was stimulated by recent articles by I.A. Taimanov [27], [28] on the Moutard-type transforms for the Dirac operators in the framework of the soliton theory in dimension  $2 + 1$ .

In the classical theory of generalized analytic functions it is usually assumed that

$$u \in L_p(D), \quad p > 2, \quad \text{if } D \text{ is bounded}, \quad (3)$$

$$u \in L_{p,2}(\mathbb{C}), \quad p > 2, \quad \text{if } D = \mathbb{C}, \quad (4)$$

where

$$\begin{aligned} L_{p,\nu}(\mathbb{C}) \text{ denotes complex-valued functions } u \text{ such that} \\ u \in L_p(D_1), \quad u_\nu \in L_p(D_1), \quad \text{where } u_\nu(z) = \frac{1}{|z|^\nu} u\left(\frac{1}{z}\right), \\ D_1 = \{z \in \mathbb{C} : |z| \leq 1\}; \end{aligned} \quad (5)$$

see [30].

On the other hand, one of the most important applications of the generalized analytic functions theory is associated with the inverse scattering in two dimensions, see [2], [8]–[12], [15], [18], [20], [24] and references therein. In addition, already in [15] it was shown that, for the case of the two-dimensional

Schrödinger equation, not only regular generalized analytic functions, where  $u$  satisfies (3) or (4), but generalized analytic functions with contour poles, also naturally arise. However, in the latter case, assumptions (3), (4) are not valid at all. It is quite likely, that the classical methods of generalized analytic functions do not admit appropriate generalizations for this case. In particular, the problem of properly solving the generalized analytic function equation (1) for  $u$  with contour poles was remaining well-known, but open.

It is in order to precise that in the framework of inverse scattering for the Schrödinger equation in two dimensions the generalized analytic function equation (1) with contour poles arises for the case when

$$u = \frac{u_{-1}}{\Delta}, \quad \psi = \frac{\psi_{-1}}{\Delta}, \quad (6)$$

where  $\Delta$  is real-valued and  $u_{-1}$ ,  $\psi_{-1}$ ,  $\Delta$  are quite regular (e.g. real-analytic) functions on  $D$ , and the pole contours are the zeroes of  $\Delta$ . In this setting  $\psi = \psi(x, z, E)$  are the Faddeev exponentially growing solutions of the Schrödinger equation in  $x \in \mathbb{R}^2$  at fixed energy  $E \in \mathbb{R}$ ,  $u$  is a particular case of the Faddeev generalized scattering data,  $\Delta$  is the modified Fredholm determinant for the Lipman-Schwinger-Faddeev integral equation for  $\psi$ , and  $z$  is a fixed-energy spectral parameter; see [7], [19], [15] and references therein.

Proceeding from the aforementioned inverse scattering motivation, we consider the generalized analytic function equation (1) with contour poles for the case when  $u$  is of the form as in (6), and when this equation has sufficiently many (more or less as in the regular case) local solutions  $\psi$  of the form as in (6). Adopting the terminology of [17] we say that in this case equation (1) is of meromorphic class. Note also that equation (1) may be of meromorphic class only if principal terms of  $u$  near pole contours satisfy solvability conditions; for simple contour poles these conditions were found in [15].

Actually, our recent works [13], [14] were motivated considerably by the aforementioned open problem of properly solving the generalized analytic function equation (1) with contour poles for the meromorphic case. In particular, in [13] we give examples of coefficients  $u$  when equation (1) is of meromorphic class and can be efficiently studied using Moutard-type transforms. In addition, we were stimulated by results of [1], [4], [16], [25], [29], [27] on efficient applications of Darboux-Crum and Moutard-type transforms to studies of some important linear ODE's and PDS's with singular coefficients.

The results of the present work can be summarized as follows:

- We give composition and inversion formulas for the simple Moutard-type transforms for the equations (1), (2); see Theorems 1, 2 in Subsection 2.2;
- We show that any equation (1) of meromorphic class with a simple contour pole can be transformed to a regular one in a neighborhood of the pole contour via an appropriate simple Moutard-type transform; see Theorem 3 in Subsection 3.4.

## 2 Moutard-type transforms for generalized analytic functions

### 2.1 Basic construction

Let

$$f_j = f_j(z) \quad \text{and} \quad f_j^+ = f_j^+(z), \quad j = 1, \dots, N, \quad (7)$$

denote a set of fixed solutions of equations (1) and (2), respectively.

Let  $\psi, \psi^+$  be arbitrary solutions of (1), (2). We define an imaginary-valued potential  $\omega_{\psi, \psi^+} = \omega_{\psi, \psi^+}(z)$  such that

$$\partial_z \omega_{\psi, \psi^+} = \psi \psi^+, \quad \partial_{\bar{z}} \omega_{\psi, \psi^+} = -\overline{\psi \psi^+} \quad \text{in } D. \quad (8)$$

We recall that this definition is self-consistent, at least, under the assumption that  $D$  is simply-connected. The integration constant is imaginary-valued and may depend on concrete situation.

As in [13], [14] we consider imaginary-valued potentials  $\omega_{f_j, f_k^+}$ ,  $j, k = 1, \dots, N$ , and we set

$$\Omega = \begin{bmatrix} \omega_{f_1, f_1^+} & \omega_{f_2, f_1^+} & \dots & \omega_{f_N, f_1^+} \\ \omega_{f_1, f_2^+} & \omega_{f_2, f_2^+} & \dots & \omega_{f_N, f_2^+} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{f_1, f_N^+} & \omega_{f_2, f_N^+} & \dots & \omega_{f_N, f_N^+} \end{bmatrix}. \quad (9)$$

Following [13], for equations (1), (2) we consider the Moutard-type transform

$$u \xrightarrow{\mathcal{M}} \tilde{u}, \quad (10)$$

$$\{\psi, \psi^+\} \xrightarrow{\mathcal{M}} \{\tilde{\psi}, \tilde{\psi}^+\}, \quad (11)$$

$$\mathcal{M} = \mathcal{M}_{u, f, f^+} = \mathcal{M}_{u, f_1, \dots, f_N, f_1^+, \dots, f_N^+} \quad (12)$$

defined as follows:

$$\tilde{u} = u + \begin{bmatrix} f_1 & \dots & f_N \end{bmatrix} \Omega^{-1} \begin{bmatrix} f_1^+ \\ \vdots \\ f_N^+ \end{bmatrix}, \quad (13)$$

$$\tilde{\psi} = \psi - \begin{bmatrix} f_1 & \dots & f_N \end{bmatrix} \Omega^{-1} \begin{bmatrix} \omega_{\psi, f_1^+} \\ \vdots \\ \omega_{\psi, f_N^+} \end{bmatrix}, \quad (14)$$

$$\tilde{\psi}^+ = \psi^+ - \begin{bmatrix} f_1^+ & \dots & f_N^+ \end{bmatrix} (\Omega^{-1})^t \begin{bmatrix} \omega_{f_1, \psi^+} \\ \vdots \\ \omega_{f_N, \psi^+} \end{bmatrix}, \quad (15)$$

where  $\psi$ ,  $\psi^+$  are formal solutions to equations (1), (2), respectively,  $\omega_{\psi, f_j^+}$  and  $\omega_{f_j, \psi^+}$  are defined as in (8);  $t$  in (15) stands for the matrix transposition.

Due to results of [13], the transformed functions  $\tilde{\psi}$ ,  $\tilde{\psi}^+$  solve the transformed generalized-analytic function equations:

$$\partial_{\bar{z}} \tilde{\psi} = \tilde{u} \overline{\tilde{\psi}} \quad \text{in } D, \quad (16)$$

$$\partial_{\bar{z}} \tilde{\psi}^+ = -\overline{\tilde{u}} \tilde{\psi}^+ \quad \text{in } D. \quad (17)$$

## 2.2 Composition and inversion of simple Moutard transforms

We say that the Moutard-type transforms (10)- (15) are simple if  $N = 1$ . In this case:

$$\tilde{\psi} = \psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}}, \quad (18)$$

$$\tilde{\psi}^+ = \psi^+ - f_1^+ \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}}, \quad (19)$$

$$\tilde{u} = u + \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}}. \quad (20)$$

**Proposition 1** *For a simple Moutard transform (18)- (20) the following formula holds:*

$$\omega_{\tilde{\psi}, \tilde{\psi}^+} = \frac{\omega_{\psi, \psi^+} \omega_{f_1, f_1^+} - \omega_{\psi, f_1^+} \omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} + c_{\tilde{\psi}, \tilde{\psi}^+}, \quad (21)$$

where  $\omega_{\tilde{\psi}, \tilde{\psi}^+}$  is defined according to (8), and  $c_{\tilde{\psi}, \tilde{\psi}^+}$  is an imaginary-valued integration constant.

Proposition 1 was announced in [14] and is proved in Section 4 of the present work.

Let  $f_1, f_2$  and  $f_1^+, f_2^+$  be some fixed solutions of equations (1) and (2), respectively, with given  $u$ . Let

$$u \xrightarrow{\mathcal{M}} \tilde{u}, \quad (22)$$

$$\{\psi, \psi^+\} \xrightarrow{\mathcal{M}} \{\tilde{\psi}, \tilde{\psi}^+\}, \quad (23)$$

$$\mathcal{M} = \mathcal{M}_2 \circ \mathcal{M}_1, \quad (24)$$

where

1.  $\mathcal{M}_1$  is the simple Moutard transform for equations (1), (2) with coefficient  $u$ , generated by  $f_1, f_1^+$  and given by formulas (18)-(20);
2.  $\mathcal{M}_2$  is the simple Moutard transform for equations (16), (17) with coefficient  $\tilde{u} = \mathcal{M}_1 u$ , given by:

$$\tilde{\psi} = \psi - \tilde{f}_2 \frac{\omega_{\tilde{\psi}, \tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}}, \quad (25)$$

$$\tilde{\psi}^+ = \psi^+ - \tilde{f}_2^+ \frac{\omega_{\tilde{f}_2, \tilde{\psi}^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}}, \quad (26)$$

$$\tilde{u} = u + \frac{\tilde{f}_2 \overline{\tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}}, \quad (27)$$

where we assume that:

- $\tilde{\psi} = \mathcal{M}_1 \psi, \tilde{\psi}^+ = \mathcal{M}_1 \psi^+, \tilde{f}_2 = \mathcal{M}_1 f_2, \tilde{f}_2^+ = \mathcal{M}_1 f_2^+;$

- $\omega_{\tilde{\psi}, \tilde{f}_2^+}$ ,  $\omega_{\tilde{f}_2, \tilde{\psi}^+}$  and  $\omega_{\tilde{f}_2, \tilde{f}_2^+}$  are given by (21) with  $c_{\tilde{\psi}, \tilde{f}_2^+}$ ,  $c_{\tilde{f}_2, \tilde{\psi}^+}$  and  $c_{\tilde{f}_2, \tilde{f}_2^+}$  equal to zero.

3. We assume that  $\omega_{f_1, f_1^+} \neq 0$ ,  $\omega_{f_2, f_2^+} \neq 0$ .

**Theorem 1** *Let  $f_1$ ,  $f_2$  and  $f_1^+$ ,  $f_2^+$  be some fixed solutions of equations (1) and (2), respectively, with given  $u$ , and let  $\mathcal{M}$  be defined as in (22)-(24). Then  $\mathcal{M}$  coincides with the Moutard transform given by formulas (10)-(15) for  $N = 2$  and generated by the initial functions  $f_1$ ,  $f_2$ ,  $f_1^+$ ,  $f_2^+$ .*

Schematically, the result of Theorem 1 can be also presented as follows:

$$\begin{array}{ccccc} \psi & \xrightarrow[\mathcal{M}_1]{\{f_1, f_1^+\}} & \tilde{\psi} & \xrightarrow[\mathcal{M}_2]{\{\tilde{f}_2, \tilde{f}_2^+\}} & \tilde{\tilde{\psi}} \\ & \searrow \{f_1, f_2, f_1^+, f_2^+\} & & & \nearrow \end{array} \quad (28)$$

Theorem 1 is proved in Section 4.

Next, in order to inverse simple Moutard transforms, we consider:

$$f_2 \equiv 0, \quad f_2^+ \equiv 0, \quad \omega_{f_2, f_1^+} = i, \quad \omega_{f_1, f_2^+} = i; \quad (29)$$

$$\tilde{f}_2 = \mathcal{M}_1 f_2 = \hat{f} = -i \frac{f_1}{\omega_{f_1, f_1^+}}, \quad \tilde{f}_2^+ = \mathcal{M}_1 f_2^+ = \hat{f}^+ = -i \frac{f_1^+}{\omega_{f_1, f_1^+}}. \quad (30)$$

In particular,  $\hat{f}$  and  $\hat{f}^+$  defined in (30) satisfy equations (16) and (17), respectively, with the coefficient  $\tilde{u}$  given by (20).

**Theorem 2** *Let  $\mathcal{M}_1$  be a simple Moutard transform defined as in (18)-(20), where  $\omega_{f_1, f_1^+} \neq 0$ . Let  $\mathcal{M}_2$  be the simple Moutard transform for equations (16), (17) with coefficient  $\tilde{u} = \mathcal{M}_1 u$ , given by (25)-(27), where  $\tilde{\psi} = \mathcal{M}_1 \psi$ ,  $\tilde{\psi}^+ = \mathcal{M}_1 \psi^+$ ,  $\tilde{f}_2 = \hat{f}$ ,  $\tilde{f}_2^+ = \hat{f}^+$ , where  $\hat{f}$  and  $\hat{f}^+$  are defined in (30). Then:*

1. The potentials  $\omega_{\tilde{\psi}, \tilde{f}_2^+}$ ,  $\omega_{\tilde{f}_2, \tilde{\psi}^+}$ ,  $\omega_{\tilde{f}_2, \tilde{f}_2^+}$  can be chosen as follows:

$$\omega_{\tilde{\psi}, \tilde{f}_2^+} = -i \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}}, \quad \omega_{\tilde{f}_2, \tilde{\psi}^+} = -i \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}}, \quad \omega_{\tilde{f}_2, \tilde{f}_2^+} = \frac{1}{\omega_{f_1, f_1^+}}; \quad (31)$$

2. Under assumptions (31), the composition  $\mathcal{M} = \mathcal{M}_2 \circ \mathcal{M}_1$  is the identical transformation.

Theorem 2 is proved in Section 4.



### 3 Removing the simplest contour pole singularity

#### 3.1 Real analytic pole contour

We consider a real analytic curve  $\Gamma \subset D$ :

$$\Gamma = \{z(\tau) : \tau \in ]it_1, it_2[ \}, \quad t_1, t_2 \in \mathbb{R}, \quad (32)$$

where:

- $z(\tau) \in D$  for  $\tau \in ]it_1, it_2[$ ,
- $z$  is (complex-valued) real-analytic on  $]it_1, it_2[$ ,
- $z(\tau'_1) \neq z(\tau'_2)$  for  $\tau'_1 \neq \tau'_2$ ,
- $dz(it)/dt \neq 0$  for  $t \in ]t_1, t_2[$ .

Here  $D$  is the domain in (1), (2).

Let

$$\mathcal{T} = \mathcal{T}_{a,b,\varepsilon} = \{\tau \in \mathbb{C} : a < \text{Im } \tau < b, |\text{Re } \tau| < \varepsilon\}, \quad (33)$$

where  $a, b, \varepsilon \in \mathbb{R}$ ,  $\varepsilon > 0$ .

As a corollary of our assumptions,  $z$  in (32) could be continued to a holomorphic bijection  $Z$ :

$$\begin{aligned} Z : \mathcal{T}_{a,b,\varepsilon} &\rightarrow D_{a,b,\varepsilon}, \quad \tau \rightarrow z(\tau), \\ Z^{-1} : D_{a,b,\varepsilon} &\rightarrow \mathcal{T}_{a,b,\varepsilon}, \quad z \rightarrow \tau(z), \end{aligned} \quad (34)$$

for some  $a, b, \varepsilon$  such that  $t_1 < a < b < t_2$ ,  $\varepsilon > 0$ , where  $D_{a,b,\varepsilon} \subset D$ .

Actually, we consider  $\Gamma$  of (32) as a pole contour for equations (1), (2).

#### 3.2 Holomorphic change of variables

In the domain  $D_{a,b,\varepsilon}$  of (34) we rewrite equations (1), (2) in variable  $\tau \in \mathcal{T}_{a,b,\varepsilon}$ . In this connection, following [14] we consider

$$u_*(\tau) = u(z(\tau)) \left| \frac{\partial z(\tau)}{\partial \tau} \right|, \quad (35)$$

$$\psi_*(\tau) = \psi(z(\tau)) \sqrt{\frac{\partial z(\tau)}{\partial \tau}}, \quad \psi_*^+(\tau) = \psi^+(z(\tau)) \sqrt{\frac{\partial z(\tau)}{\partial \tau}}, \quad (36)$$

where  $u(z)$ ,  $\psi(z)$ ,  $\psi^+(z)$  are the same that in equations (1), (2),  $z(\tau)$  is the same that in (34),  $\tau \in \mathcal{T}_{a,b,\varepsilon}$ . Then (see [14]):

$$\partial_{\bar{\tau}}\psi_* = u_*\bar{\psi}_* \text{ in } \mathcal{T}_{a,b,\varepsilon}, \quad (37)$$

$$\partial_{\bar{\tau}}\psi_*^+ = -\bar{u}_*\bar{\psi}_*^+ \text{ in } \mathcal{T}_{a,b,\varepsilon}. \quad (38)$$

In addition, we have (see [14]):

$$\mathcal{M}_{u_*,f_*,f_*^+} \circ Z^{-1} = Z^{-1} \circ \mathcal{M}_{u,f,f^+}, \quad (39)$$

where:

- $Z^{-1}$  is considered as a map of the conjugate pairs of equations (1), (2) in  $D_{a,b,\varepsilon}$  into the conjugate pairs of equations (37), (38) in  $\mathcal{T}_{a,b,\varepsilon}$  and is defined according to (35), (36);
- $\mathcal{M}_{u,f,f^+}$  for (1), (2) in  $D_{a,b,\varepsilon}$  and  $\mathcal{M}_{u_*,f_*,f_*^+}$  for (37), (38) in  $\mathcal{T}_{a,b,\varepsilon}$  are defined according to formulas (10)-(15), where  $u_*$ ,  $f_* = \{f_{1,*}, \dots, f_{N,*}\}$ ,  $f_*^+ = \{f_{1,*}^+, \dots, f_{N,*}^+\}$  are defined according to (35), (36), and

$$\omega_{\psi_*,\psi_*^+}(\tau) = \omega_{\psi,\psi^+}(z(\tau)), \quad (40)$$

for all involved potentials.

In the framework of the Moutard transform approach, using the commutativity relation (39) we reduce local studies of equations (1), (2) with contour pole at  $\Gamma$  in (32) to the case of contour pole at the straight line

$$\Gamma_* = \{\tau \in \mathcal{T}_{a,b,\varepsilon} : \operatorname{Re} \tau = 0\}, \quad (41)$$

where  $\mathcal{T}_{a,b,\varepsilon}$  is defined as in (33), (34).

**Remark 1** *We recall that, in view of formulas (35), (36), the generalized analytic functions  $\psi$ ,  $\psi^+$  of (1), (2) can be treated as spinors, i.e. differential forms of type  $(\frac{1}{2}, 0)$ , and  $u$  can be treated as differential form of type  $(\frac{1}{2}, \frac{1}{2})$ ; see [14]. These forms can be written as:*

$$u = u(z)\sqrt{dzd\bar{z}}, \quad \psi = \psi(z)\sqrt{dz}, \quad \psi^+ = \psi^+(z)\sqrt{dz}. \quad (42)$$

### 3.3 Constraints on the meromorphic class coefficients at singularity

We consider equations (37), (38) in  $\mathcal{T} = \mathcal{T}_{a,b,\varepsilon}$  defined by (33) for the case of simplest pole at  $\Gamma_*$  defined by (41). We write  $\tau = x + iy$ ,  $\bar{\tau} = x - iy$ .

We assume that

$$u_*(\tau) = e^{2i\phi(y)} \sum_{j=-n}^{+\infty} r_j(y) x^j \quad \text{in } \mathcal{T}, \quad (43)$$

where  $\phi, r_j$  are quite regular functions on the interval  $]a, b[ \approx \Gamma_*$ , and  $\phi, r_{-n}$  are real-valued,  $n \in \mathbb{N}$ . For this case we consider solutions  $\psi_*, \psi_*^+$  of (37), (38) near  $\Gamma_*$  of the following form:

$$\psi_* = \sum_{j=-n'}^{+\infty} \alpha_j(y) x^j, \quad \psi_*^+ = \sum_{j=-n''}^{+\infty} \alpha_j^+(y) x^j, \quad n', n'' \in \mathbb{N}, \quad (44)$$

where  $\alpha_j, \alpha_j^+$  are quite regular on  $]a, b[$  and  $\alpha_{-n'}, \alpha_{-n''}^+$  are not-zero almost everywhere at  $]a, b[$ .

Note that in this section we consider  $u_*, \psi_*, \psi_*^+$  in formulas (43), (44) as formal power series in variable  $x$ .

**Lemma 1** *Assume that equation (37) with coefficient  $u_*$  as in (43) has, at least, one solution  $\psi_*$  of the form as in (44). Then  $n = 1$  and  $|r_{-1}(y)| \equiv n'/2$  in (43).*

Lemma 1 follows from formal substitutions of (43), (44) into (37), (38).

In the present article we restrict ourselves to the simplest (but generic) case when  $n' = n'' = 1$ . In this case, without loss of generality, we can assume that  $r_{-1} = -1/2$ , adding  $\pi/2$  to the phase  $\phi$ , if necessary.

Then, as a corollary of results of [15], equation (37) is of meromorphic class, at least formally, if and only if:

$$\operatorname{Re} r_0(y) \equiv 0, \quad y \in ]a, b[, \quad (45)$$

$$\operatorname{Im} r_1(y) = \frac{1}{2} \frac{d^2 \phi(y)}{dy^2}, \quad y \in ]a, b[. \quad (46)$$

For completeness of exposition, this result is proved in Section 5.

Here, belonging of equation (37) to meromorphic class means that equation (37) has local solutions  $\psi_*$  near  $\Gamma_* \subset \mathcal{T}$  of the form (44) with  $n' = 1$  parametrised by two real-valued functions (one complex-valued function) on  $\Gamma_*$ , i.e., roughly speaking, equation (37) has as many local solutions  $\psi_*$  near  $\Gamma_*$  as in the regular case.

Actually, equation (37) and formulas (43), (44) for  $u_*$ ,  $\psi_*$  with  $n = 1$ ,  $n' = 1$ ,  $r_{-1} = -1/2$  imply that

$$\operatorname{Im} e^{-i\phi(y)} \alpha_{-1}(y) = 0, \quad y \in ]a, b[. \quad (47)$$

In addition, under conditions (45), (46), the solutions  $\psi_*$  of (37), (44) are parametrised by  $\beta_{-1}(y)$  and  $\operatorname{Im} \beta_1(y)$  on  $]a, b[$ , where  $\alpha_j(y) = e^{i\phi(y)} \beta_j(y)$ ; see Section 5.

Finally, one can see that the meromorphic class conditions (45), (46) for equation (37) imply the related meromorphic class conditions for the conjugate equation (38).

### 3.4 Moutard transform to the regular case

We consider equations (37), (38) for the case when

$$u_* = e^{2i\phi(y)} \left( -\frac{1}{2x} + r_0(y) + r_1(y)x + O(x^2) \right) \quad \text{in } \mathcal{T} \cup \partial\mathcal{T}, \quad (48)$$

where

$$\begin{aligned} \phi & \text{ is real-valued, } \phi \in C^2([a, b]), \\ u_* + \frac{e^{2i\phi(y)}}{2x} & \in C^1(\mathcal{T} \cup \partial\mathcal{T}), \\ r_0, r_1 & \text{ satisfy (45), (46),} \end{aligned} \quad (49)$$

$\mathcal{T}$  is defined by (33).

We assume that equations (37), (38) have some solutions  $f_*$ ,  $f_*^+$  such that

$$\begin{aligned} f_* &= e^{i\phi(y)} \left( \frac{\beta_{-1}(y)}{x} + \beta_0(y) + O(x) \right), \\ \beta_{-1} & \text{ is real-valued, } \beta_{-1} > 0, \quad \beta_{-1} \in C^1([a, b]), \\ f_* - \frac{e^{i\phi(y)} \beta_{-1}(y)}{x} & \in C^1(\mathcal{T} \cup \partial\mathcal{T}), \end{aligned} \quad (50)$$

$$\begin{aligned}
f_*^+ &= e^{-i(\phi(y)+\pi/2)} \left( \frac{\beta_{-1}^+(y)}{x} + \beta_0^+(y) + O(x) \right), \\
\beta_{-1}^+ &\text{ is real-valued, } \beta_{-1}^+ > 0, \quad \beta_{-1}^+ \in C^1([a, b]), \\
f_*^+ - \frac{e^{-i(\phi(y)+\pi/2)} \beta_{-1}^+(y)}{x} &\in C^1(\mathcal{T} \cup \partial\mathcal{T}),
\end{aligned} \tag{51}$$

where  $\partial_x O(x) = O(1)$ ,  $\partial_y O(x) = O(x)$ .

Note that from point of view of formal considerations of Subsection 3.3 such solutions  $f_*$ ,  $f_*^+$  always exist.

**Theorem 3** *Let  $u_*$  satisfy (48), (49) and equations (37), (38) have some solutions  $f_*$ ,  $f_*^+$  satisfying (50), (51). Let  $\omega_{f_*, f_*^+}$  be some potential defined according to (8). Then*

$$\tilde{u}_* = \mathcal{M}_{u_*, f_*, f_*^+} u_* = O(1) \quad \text{in } \mathcal{T}_{a,b,\delta} \cup \partial\mathcal{T}_{a,b,\delta} \tag{52}$$

for some  $\delta \in ]0, \varepsilon[$ , where  $\mathcal{M}_{u_*, f_*, f_*^+}$  is defined according to formulas (10)-(15) for  $N = 1$ .

*Theorem 3 is proved in Section 6.*

The point is that the results of Theorems 2, 3 and the commutativity formula (39) reduce local studies of equations (1), (2) near the simplest contour pole singularity to the regular case.

## 4 Proofs of Proposition 1 and Thereoms 1-2

### 4.1 Proof of Proposition 1

Let  $\omega_{\tilde{\psi}, \tilde{\psi}^+}$  be given by (21). Then it is sufficient to show that

$$\partial_z \omega_{\tilde{\psi}, \tilde{\psi}^+} = \tilde{\psi} \tilde{\psi}^+, \quad \partial_{\bar{z}} \omega_{\tilde{\psi}, \tilde{\psi}^+} = -\overline{\tilde{\psi} \tilde{\psi}^+}. \tag{53}$$

Using the definitions of  $\omega_{\psi, \psi^+}$ ,  $\omega_{\psi, f_1^+}$ ,  $\omega_{f_1, \psi^+}$ ,  $\omega_{f_1, f_1^+}$ , and  $\tilde{\psi}$ ,  $\tilde{\psi}^+$  we have:

$$\begin{aligned}
\partial_z \omega_{\tilde{\psi}, \tilde{\psi}^+} &= \partial_z \left( \frac{\omega_{\psi, \psi^+} \omega_{f_1, f_1^+} - \omega_{\psi, f_1^+} \omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} + c_{\tilde{\psi}, \tilde{\psi}^+} \right) = \\
&= \partial_z \omega_{\psi, \psi^+} - \partial_z \omega_{\psi, f_1^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} - \partial_z \omega_{f_1, \psi^+} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} + \partial_z \omega_{f_1, f_1^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} =
\end{aligned}$$

$$\begin{aligned}
&= \psi\psi^+ - \psi f_1^+ \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} - f_1 \psi^+ \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} + \psi\psi^+ \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = \\
&= \left( \psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left( \psi^+ - f_1^+ \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \right) = \tilde{\psi}\tilde{\psi}^+; \\
&\partial_{\bar{z}}\omega_{\tilde{\psi}, \tilde{\psi}^+} = \partial_{\bar{z}} \left( \frac{\omega_{\psi, \psi^+} \omega_{f_1, f_1^+} - \omega_{\psi, f_1^+} \omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} + c_{\tilde{\psi}, \tilde{\psi}^+} \right) = \\
&= \partial_{\bar{z}}\omega_{\psi, \psi^+} - \partial_{\bar{z}}\omega_{\psi, f_1^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} - \partial_{\bar{z}}\omega_{f_1, \psi^+} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} + \partial_z\omega_{f_1, f_1^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = \\
&= -\overline{\psi\psi^+} + \overline{\psi} f_1^+ \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} + \overline{f_1} \psi^+ \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} - \overline{\psi\psi^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = \\
&= - \left( \overline{\psi} - \overline{f_1} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left( \overline{\psi^+} - \overline{f_1^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \right) = -\overline{\tilde{\psi}\tilde{\psi}^+}.
\end{aligned}$$

Thus, the proof of Proposition 1 is completed.

## 4.2 Proof of Theorem 1

Due to formula (21) for  $\omega_{\tilde{f}_2, \tilde{f}_2^+}$  with  $c_{\tilde{f}_2, \tilde{f}_2^+} = 0$  and due to the assumptions that  $\omega_{f_1, f_1^+} \neq 0$ ,  $\omega_{\tilde{f}_2, \tilde{f}_2^+} \neq 0$ , we have:

$$\det \Omega = \omega_{f_1, f_1^+} \omega_{\tilde{f}_2, \tilde{f}_2^+} \neq 0, \quad (54)$$

where  $\Omega$  is defined according to (9) for  $N = 2$ .

Due to the definition of  $\mathcal{M}$  according to (22)-(24), using (21) for  $\omega_{\tilde{\psi}, \tilde{\psi}^+} = \omega_{\tilde{f}_2, \tilde{f}_2^+}$ ,  $\omega_{\tilde{\psi}, \tilde{f}_2^+}$  with  $c_{\tilde{\psi}, \tilde{\psi}^+} = 0$  and using (54), we obtain:

$$\begin{aligned}
\tilde{\psi} &= \tilde{\psi} - \tilde{f}_2 \frac{\omega_{\tilde{\psi}, \tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = \psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} - \\
&- \left( f_2 - f_1 \frac{\omega_{f_2, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left( \frac{\omega_{\psi, f_2^+} \omega_{f_1, f_1^+} - \omega_{\psi, f_1^+} \omega_{f_1, f_2^+}}{\omega_{f_1, f_1^+}} \right) \left( \frac{\omega_{f_2, f_2^+} \omega_{f_1, f_1^+} - \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}}{\omega_{f_1, f_1^+}} \right)^{-1} = \\
&= \psi - f_1 \frac{1}{\omega_{f_1, f_1^+} \omega_{f_2, f_2^+} - \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}} \times
\end{aligned}$$

$$\begin{aligned}
& \times \left( \omega_{\psi, f_1^+} \omega_{f_2, f_2^+} - \frac{\omega_{\psi, f_1^+} \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}}{\omega_{f_1, f_1^+}} - \omega_{\psi, f_2^+} \omega_{f_2, f_1^+} + \frac{\omega_{f_2, f_1^+} \omega_{\psi, f_1^+} \omega_{f_1, f_2^+}}{\omega_{f_1, f_1^+}} \right) - \\
& - f_2 \frac{1}{\omega_{f_1, f_1^+} \omega_{f_2, f_2^+} - \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}} \left( \omega_{\psi, f_2^+} \omega_{f_1, f_1^+} - \omega_{\psi, f_1^+} \omega_{f_1, f_2^+} \right) = \\
& = \psi - \begin{bmatrix} f_1 & f_2 \end{bmatrix} \frac{1}{\omega_{f_1, f_1^+} \omega_{f_2, f_2^+} - \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}} \begin{bmatrix} \omega_{\psi, f_1^+} \omega_{f_2, f_2^+} - \omega_{\psi, f_2^+} \omega_{f_2, f_1^+} \\ \omega_{\psi, f_2^+} \omega_{f_1, f_1^+} - \omega_{\psi, f_1^+} \omega_{f_1, f_2^+} \end{bmatrix}. \tag{55}
\end{aligned}$$

Taking into account that

$$\begin{aligned}
\Omega &= \begin{bmatrix} \omega_{f_1, f_1^+} & \omega_{f_2, f_1^+} \\ \omega_{f_1, f_2^+} & \omega_{f_2, f_2^+} \end{bmatrix}, \quad \Omega^{-1} = \frac{1}{\omega_{f_1, f_1^+} \omega_{f_2, f_2^+} - \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}} \begin{bmatrix} \omega_{f_2, f_2^+} & -\omega_{f_2, f_1^+} \\ -\omega_{f_1, f_2^+} & \omega_{f_1, f_1^+} \end{bmatrix}, \\
\Omega^{-1} \begin{bmatrix} \omega_{\psi, f_1^+} \\ \omega_{\psi, f_2^+} \end{bmatrix} &= \frac{1}{\omega_{f_1, f_1^+} \omega_{f_2, f_2^+} - \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}} \begin{bmatrix} \omega_{\psi, f_1^+} \omega_{f_2, f_2^+} - \omega_{\psi, f_2^+} \omega_{f_2, f_1^+} \\ \omega_{\psi, f_2^+} \omega_{f_1, f_1^+} - \omega_{\psi, f_1^+} \omega_{f_1, f_2^+} \end{bmatrix},
\end{aligned}$$

formula (55) can be rewritten as:

$$\tilde{\psi} = \psi - \begin{bmatrix} f_1 & f_2 \end{bmatrix} \Omega^{-1} \begin{bmatrix} \omega_{\psi, f_1^+} \\ \omega_{\psi, f_2^+} \end{bmatrix}. \tag{56}$$

One can see that (56) coincides with formula (14) for  $N = 2$ .

The computations for  $\tilde{\psi}^+$  are similar.

In additions, due to the definition of  $\mathcal{M}$  according to (22)-(24), using (21) for  $\omega_{\tilde{f}_2, \tilde{f}_2^+}$  with  $c_{\tilde{f}_2, \tilde{f}_2^+} = 0$  and using (54), we obtain:

$$\begin{aligned}
\tilde{u} &= \tilde{u} + \frac{\tilde{f}_2 \overline{\tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = u + \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}} + \frac{\tilde{f}_2 \overline{\tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = \\
&= u + \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}} + \left( f_2 - f_1 \frac{\omega_{f_2, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left( \overline{f_2^+} - \overline{f_1^+} \frac{\omega_{f_1, f_2^+}}{\omega_{f_1, f_1^+}} \right) \frac{\omega_{f_1, f_1^+}}{\omega_{f_1, f_1^+} \omega_{f_2, f_2^+} - \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}} = \\
&= u + \frac{f_1 \overline{f_1^+} \omega_{f_2, f_2^+} - f_2 \overline{f_1^+} \omega_{f_1, f_2^+} - f_1 \overline{f_2^+} \omega_{f_2, f_1^+} + f_2 \overline{f_2^+} \omega_{f_1, f_1^+}}{\omega_{f_1, f_1^+} \omega_{f_2, f_2^+} - \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}} = \\
&= u + \frac{1}{\omega_{f_1, f_1^+} \omega_{f_2, f_2^+} - \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} \omega_{f_2, f_2^+} & -\omega_{f_2, f_1^+} \\ -\omega_{f_1, f_2^+} & \omega_{f_1, f_1^+} \end{bmatrix} \begin{bmatrix} \overline{f_1^+} \\ \overline{f_2^+} \end{bmatrix}. \tag{57}
\end{aligned}$$

One can see that (57) coincides with formula (13) for  $N = 2$ .

This completes the proof of Theorem 1.

### 4.3 Proof of Theorem 2

First, we check that  $\omega_{\tilde{f}_2, \tilde{\psi}^+}$ ,  $\omega_{\tilde{\psi}, \tilde{f}_2^+}$ ,  $\omega_{\tilde{f}_2, \tilde{f}_2^+}$  defined in (31) are the potentials for the pairs  $\{\hat{f}, \tilde{\psi}^+\}$ ,  $\{\tilde{\psi}, \hat{f}^+\}$ ,  $\{\hat{f}, \hat{f}^+\}$ :

$$\begin{aligned}\partial_z \omega_{\tilde{\psi}, \tilde{f}_2^+} &= -i \partial_z \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = -i \frac{\psi f_1^+}{\omega_{f_1, f_1^+}} + i \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}^2} f_1 f_1^+ = \\ &= \left( \psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left( \frac{-i f_1^+}{\omega_{f_1, f_1^+}} \right) = \tilde{\psi} \hat{f}^+, \\ \partial_{\bar{z}} \omega_{\tilde{\psi}, \tilde{f}_2^+} &= -i \partial_{\bar{z}} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = i \frac{\bar{\psi} \bar{f}_1^+}{\omega_{f_1, f_1^+}} - i \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}^2} \bar{f}_1 \bar{f}_1^+ = \\ &= - \left( \bar{\psi} - \bar{f}_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left( \frac{-i \bar{f}_1^+}{\omega_{f_1, f_1^+}} \right) = -\tilde{\psi} \hat{f}^+;\end{aligned}\tag{58}$$

the calculations for  $\omega_{\tilde{f}_2, \tilde{\psi}^+}$  are analogous to the calculations for  $\omega_{\tilde{\psi}, \tilde{f}_2^+}$  ;

$$\begin{aligned}\partial_z \omega_{\tilde{f}_2, \tilde{f}_2^+} &= \partial_z \frac{1}{\omega_{f_1, f_1^+}} = - \frac{\partial_z \omega_{f_1, f_1^+}}{\omega_{f_1, f_1^+}^2} = - \frac{f_1 f_1^+}{\omega_{f_1, f_1^+}^2} = \hat{f} \hat{f}^+, \\ \partial_{\bar{z}} \omega_{\tilde{f}_2, \tilde{f}_2^+} &= \partial_{\bar{z}} \frac{1}{\omega_{f_1, f_1^+}} = - \frac{\partial_{\bar{z}} \omega_{f_1, f_1^+}}{\omega_{f_1, f_1^+}^2} = \frac{\bar{f}_1 \bar{f}_1^+}{\omega_{f_1, f_1^+}^2} = -\bar{\hat{f}} \bar{\hat{f}}^+.\end{aligned}\tag{59}$$

Here, we used also that all potentials  $\omega_{\psi, \psi^+}$  are pure imaginary.

Second, we calculate the transform  $\mathcal{M}_2 \circ \mathcal{M}_1$ :

$$\begin{aligned}\tilde{\psi} &= \tilde{\psi} - \hat{f} \frac{\omega_{\tilde{\psi}, \tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = \psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} - \left( -i \frac{f_1}{\omega_{f_1, f_1^+}} \right) \frac{-i \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}}}{\frac{1}{\omega_{f_1, f_1^+}}} = \\ &= \psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} + f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = \psi;\end{aligned}$$

the calculations for  $\tilde{\psi}^+$  are analogous to the calculations for  $\tilde{\psi}$ ;

$$\tilde{u} = \tilde{u} + \frac{\hat{f} \hat{f}^+}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = u + \frac{f_1 \bar{f}_1^+}{\omega_{f_1, f_1^+}} + \frac{\hat{f} \hat{f}^+}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = u + \frac{f_1 \bar{f}_1^+}{\omega_{f_1, f_1^+}} + \frac{-i \frac{f_1}{\omega_{f_1, f_1^+}} \left[ -i \frac{f_1^+}{\omega_{f_1, f_1^+}} \right]}{\frac{1}{\omega_{f_1, f_1^+}}} =$$



$$= u + \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}} - \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}} = u. \quad (60)$$

This completes the proof of Theorem 2.

## 5 Proof of meromorphic class conditions

We consider equation (37) and formulas (43), (44) for  $u_*$ ,  $\psi_*$ , where  $n = 1$ ,  $r_{-1} = -1/2$ ,  $n' = 1$ . In this case formulas (43), (44) can be written as:

$$u_* = e^{2i\phi(y)} \left( -\frac{1}{2x} + r_o(y) + r_1(y)x + r_2(y)x^2 + \dots \right), \quad (61)$$

$$\psi_* = e^{i\phi(y)} \left( \frac{\beta_{-1}(y)}{x} + \beta_0(y) + \beta_1(y)x + \beta_2(y)x^2 + \dots \right). \quad (62)$$

We substitute (61), (62) into (37), and we use that

$$\begin{aligned} 2\partial_{\bar{\tau}}\psi_* &= e^{i\phi(y)} \left( -\frac{\beta_{-1}(y)}{x^2} + \frac{i\beta'_{-1}(y) - \phi'(y)\beta_{-1}(y)}{x} + \right. \\ &\quad \left. + \sum_{k=0}^{+\infty} [i\beta'_k(y) - \phi'(y)\beta_k(y) + (k+1)\beta_{k+1}(y)] x^k \right), \end{aligned} \quad (63)$$

$$\begin{aligned} 2u_*\psi_* &= e^{i\phi(y)} \left( -\frac{\overline{\beta_{-1}(y)}}{x^2} + \frac{-\overline{\beta_0(y)} + 2r_0(y)\overline{\beta_{-1}(y)}}{x} + \right. \\ &\quad \left. + \sum_{k=0}^{+\infty} \left[ -\overline{\beta_{k+1}(y)} + 2r_{k+1}(y)\overline{\beta_{-1}(y)} + 2 \sum_{l=0}^k r_l(y)\overline{\beta_{k-l}(y)} \right] x^k \right). \end{aligned} \quad (64)$$

From this point and till the end of the proof ' denotes  $\partial_y$ .

Collecting the terms at  $x^k$ ,  $k = -2, -1, 0, 1, 2, \dots$ , we obtain:

$$\overline{\beta_{-1}(y)} = \beta_{-1}(y) \quad \text{for } k = -2, \quad (65)$$

$$\overline{\beta_0(y)} = -i\beta'_{-1}(y) + \phi'(y)\beta_{-1}(y) + 2r_0(y)\overline{\beta_{-1}(y)} \quad \text{for } k = -1, \quad (66)$$

$$\begin{aligned} \overline{\beta_1(y)} + \beta_1(y) = & -i\beta'_0(y) + \phi'(y)\beta_0(y) + 2r_1(y)\overline{\beta_{-1}(y)} + \\ & + 2r_0(y)\overline{\beta_0(y)} \quad \text{for } k = 0, \end{aligned} \quad (67)$$

$$\begin{aligned} \overline{\beta_{k+1}(y)} + (k+1)\beta_{k+1}(y) = & -i\beta'_k(y) + \phi'(y)\beta_k(y) + \\ & + 2r_{k+1}(y)\overline{\beta_{-1}(y)} + 2\sum_{l=0}^k r_l(y)\overline{\beta_{k-l}(y)} \quad \text{for } k \geq 1. \end{aligned} \quad (68)$$

One can see that: relation (65) coincides with (47); relation (66) can be considered as a formula for finding  $\beta_0$ ; relations (68) can be considered as recursion relations for finding  $\beta_j$ ,  $j \geq 2$ . In addition, the real part of (67) can be considered as a formula for finding  $\text{Re } \beta_1$ , whereas the imaginary part of (67) can be rewritten as

$$\text{Im} \left[ -i\beta'_0(y) + \phi'(y)\beta_0(y) + 2r_1(y)\overline{\beta_{-1}(y)} + 2r_0(y)\overline{\beta_0(y)} \right] = 0. \quad (69)$$

Actually, relations (65), (66), (69) yield the solvability constraints on  $\phi$ ,  $r_0$ ,  $r_1$ . Substituting (65), (66) into (69) we obtain:

$$\begin{aligned} \text{Im} \left[ i\phi'(y)\beta'_{-1}(y) + ((\phi'(y))^2 + 2\phi'(y)\overline{r_0(y)})\beta_{-1}(y) + 2r_1(y)\beta_{-1}(y) - \right. \\ \left. - 2ir_0(y)\beta'_{-1}(y) + (2r_0(y)\phi'(y) + 4(r_0(y))^2)\beta_{-1}(y) + \beta''_{-1}(y) - \right. \\ \left. - i(\phi'(y) + 2\overline{r_0(y)})\beta'_{-1}(y) - i(\phi''(y) + 2\overline{r'_0(y)})\beta_{-1}(y) \right] = 0. \end{aligned} \quad (70)$$

In turn, (70) can be rewritten as:

$$\begin{aligned} \text{Im} \left[ \beta''_{-1}(y) - 2i[r_0(y) + \overline{r_0(y)}]\beta'_{-1}(y) + [(\phi'(y))^2 + \right. \\ \left. + 2\phi'(y)(\overline{r_0(y)} + r_0(y)) + 4(r_0(y))^2 + 2r_1(y) - i\phi''(y) - 2i\overline{r'_0(y)}]\beta_{-1}(y) \right] = 0. \end{aligned} \quad (71)$$

In addition, taking into account that  $\phi$ ,  $\beta_{-1}$  are real-valued, we simplify (71) as follows:

$$\text{Im} \left[ -2i[r_0(y) + \overline{r_0(y)}]\beta'_{-1}(y) + [4(r_0(y))^2 + 2r_1(y) - i\phi''(y) - 2i\overline{r'_0(y)}]\beta_{-1}(y) \right] = 0. \quad (72)$$

One can see that (72) is fulfilled for all sufficiently regular real-valued  $\beta_{-1}$  if and only if (45), (46) are fulfilled.

Finally, using (65)-(68) under conditions (45), (46), one can see that all  $\psi_*$  of (62) satisfying (37) with  $u_*$  of 61) are parametrised by  $\beta_{-1}$  and  $\text{Im } \beta_1$ .

This completes the proof.

## 6 Proof of Theorem 3

Substituting (48), (50), (51) into (37), (38) we obtain:

$$\beta_0(y) = i(\beta_{-1}(y))' + (\phi'(y) - 2r_0(y))\beta_{-1}(y), \quad y \in [a, b], \quad (73)$$

$$\beta_0^+(y) = i(\beta_{-1}^+(y))' + (-\phi'(y) + 2r_0(y))\beta_{-1}^+(y), \quad y \in [a, b]. \quad (74)$$

As in Section 5 we assume that  $'$  denotes  $\partial_y$ .

Using (50), (51) and (73), (74) we obtain:

$$\begin{aligned} f_* f_*^+ &= e^{-i\pi/2} \left( \frac{\beta_{-1}(y)\beta_{-1}^+(y)}{x^2} + \frac{\beta_0(y)\beta_{-1}^+(y) + \beta_{-1}(y)\beta_0^+(y)}{x} + O(1) \right) = \\ &= -i \frac{\beta_{-1}(y)\beta_{-1}^+(y)}{x^2} + \frac{1}{x} \left( \beta_{-1}(y)\beta_{-1}^+(y) \right)' + O(1) \quad \text{in } \mathcal{T} \cup \partial\mathcal{T}. \end{aligned} \quad (75)$$

Next, equation (8) can be rewritten as:

$$\partial_x \omega_{\psi, \psi^+} = 2i \text{Im} \left( \psi \psi^+ \right), \quad \partial_y \omega_{\psi, \psi^+} = 2i \text{Re} \left( \psi \psi^+ \right). \quad (76)$$

As a corollary of (75), (76) and the property that  $\beta_1, \beta_1^+$  are real-valued, we have:

$$\partial_x \omega_{f_*, f_*^+} = \frac{-2i\beta_{-1}(y)\beta_{-1}^+(y)}{x^2} + O(1), \quad \partial_y \omega_{f_*, f_*^+} = \frac{2i(\beta_{-1}(y)\beta_{-1}^+(y))'}{x} + O(1); \quad (77)$$

$$\omega_{f_*, f_*^+} = \frac{2i}{x} \beta_{-1}(y)\beta_{-1}^+(y) + O(1) \quad \text{in } \mathcal{T} \cup \partial\mathcal{T}. \quad (78)$$

**Remark 2** Note that

$$\text{res} \Big|_{x=0} \partial_x (\omega_{f_*, f_*^+}) = 0.$$

for any fixed  $y = y_0$ .

Using also the strict positivity of  $\beta_{-1}\beta_{-1}^+$  we obtain:

$$\frac{1}{\omega_{f_*, f_*^+}} = -\frac{ix}{2\beta_{-1}(y)\beta_{-1}^+(y)} + O(x^2) \quad \text{in } \mathcal{T}_{a,b,\delta} \cup \partial\mathcal{T}_{a,b,\delta}, \quad (79)$$

for some  $\delta \in ]0, \varepsilon[$ .

Note also that

$$f_* \overline{f_*^+} = ie^{2i\phi(y)} \frac{\beta_{-1}(y)\beta_{-1}^+(y)}{x^2} + O\left(\frac{1}{x}\right) \quad \text{in } \mathcal{T} \cup \partial\mathcal{T}. \quad (80)$$

Finally, due to (20), (48), (79), (80):

$$\begin{aligned} \tilde{u}_* &= u_* + \frac{f_* \overline{f_*^+}}{\omega_{f_*, f_*^+}} = e^{2i\phi(y)} \left( -\frac{1}{2x} + r_0(y) + O(x) \right) + e^{2i\phi(y)} \frac{1}{2x} + O(1) = \\ &= O(1) \quad \text{in } \mathcal{T}_{a,b,\delta} \cup \partial\mathcal{T}_{a,b,\delta}. \end{aligned} \quad (81)$$

Theorem 3 is proved.

## References

- [1] V.A. Arkad'ev, A.K. Pogrebkov, M.K. Polivanov, Singular solutions of the KdV equation and the inverse scattering method, *Journal of Soviet Mathematics*, **31**(6) (1985), 3264-3279.
- [2] R. Beals, R.R. Coifman, The spectral problem for the Davey-Stewartson and Ishimori hierarchies, In: "Nonlinear evolution equations: integrability and spectral methods", Proc. Workshop, Como/Italy 1988, Proc. Nonlinear Sci., (1990), pp. 15-23.
- [3] L. Bers, *Theory of pseudo-analytic functions*, Courant Institute of Mathematical Sciences, New York University, Institute for Mathematics and Mechanics, 1953, 187 pages.
- [4] M.M. Crum, Associated Sturm-Liouville systems, *Quart. J. Math. Oxford Ser. (2)* **6** (1955), 121-127.
- [5] A. Doliwa, P. Grinevich, M. Nieszporski, P.M. Santini, Integrable lattices and their sublattices: From the discrete Moutard (discrete Cauchy-Riemann) 4-point equation to the self-adjoint 5-point scheme, *J. Math. Phys.*, **48**(1) (2007), 013513.

- [6] J.J. Duistermaat, F.A. Grünbaum, Differential equations in the spectral parameter, *Comm. Math. Phys.*, **103**(2) (1986), 177-240.
- [7] L.D. Faddeev, Inverse problem of quantum scattering theory. II, *Journal of Soviet Mathematics*, **5**(3) (1976), 334–396.
- [8] A.S. Fokas, L.-Y. Sung, On the solvability of the N-wave, Davey-Stewartson and Kadomtsev-Petviashvili equations, *Inverse Problems*, **8**(5) (1992), 673-708.
- [9] P.G. Grinevich, The scattering transform for the two-dimensional Schrödinger operator with a potential that decreases at infinity at fixed nonzero energy, *Russian Math. Surveys*, **55**(6) (2000), 1015-1083.
- [10] P.G. Grinevich, R.G. Novikov, Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials, *Commun. Math. Phys.*, **174** (1995), 409-446.
- [11] P.G. Grinevich, R.G. Novikov, Faddeev eigenfunctions for point potentials in two dimensions, *Physics Letters A*, **376** (2012), 1102-1106.
- [12] P.G. Grinevich, R.G. Novikov, Faddeev eigenfunctions for multipoint potentials, *Eurasian Journal of Mathematical and Computer Applications*, **1**(2) (2013), 76-91.
- [13] P.G. Grinevich, R.G. Novikov, Moutard transform for generalized analytic functions, *Journal of Geometric Analysis*, DOI 10.1007/s12220-015-9657-8, arXiv:1510.08764.
- [14] P.G. Grinevich, R.G. Novikov, Generalized analytic functions, Moutard-type transforms and holomorphic maps, arXiv:1512.00343.
- [15] P.G. Grinevich, S.P. Novikov, Two-dimensional “inverse scattering problem” for negative energies and generalized-analytic functions. 1. Energies below the ground state, *Functional Analysis and Its Applications*, **22**(1) (1988), 19-27.
- [16] P.G. Grinevich, S.P. Novikov, Singular soliton operators and indefinite metrics, *Bulletin of the Brazilian Mathematical Society, New Series*, **44**(4) (2013), 809-840.

- [17] P.G. Grinevich, S.P. Novikov, Spectrally meromorphic operators and nonlinear systems, *Russian Mathematical Surveys*, **69**(5) (2014), 924-926.
- [18] A.V. Kazeykina, A large-time asymptotics for the solution of the Cauchy problem for the Novikov-Veselov equation at negative energy with non-singular scattering data, *Inverse Problems*, **28**(5) (2012), 055017, 21 pp.
- [19] G.M. Henkin, R.G. Novikov, The  $\bar{\partial}$ -equation in the multidimensional inverse scattering problem, *Russian Math. Surveys* **42**(3) (1987), 109-180.
- [20] E.L. Lakshtanov, R.G. Novikov, B.R. Vainberg, A global Riemann-Hilbert problem for two-dimensional inverse scattering at fixed energy, arXiv:1509.06495.
- [21] V.B. Matveev, M.A. Salle, *Darboux transformations and solitons*, Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1991.
- [22] T.F. Moutard, Sur la construction des équations de la forme  $\frac{1}{z} \frac{\partial^2 z}{\partial x \partial y} = \lambda(x, y)$  qui admettent une intégrale générale explicite, *J. École Polytechnique*, **45** (1878), 1-11.
- [23] J.J.C. Nimmo, W.K. Schief, Superposition principles associated with the Moutard transformation: an integrable discretization of a 2+1-dimensional sine-Gordon system, *Proc. R. Soc. London A*, **453** (1997), 255-279.
- [24] R.G. Novikov, The inverse scattering problem on a fixed energy level for the two-dimensional Schrödinger operator, *J. Funct. Anal.* **103**(2) (1992), 409-463.
- [25] R.G. Novikov, I.A. Taimanov, The Moutard transformation and two-dimensional multipoint delta-shaped potentials, *Russian Math. Surveys*, **68**(5) (2013), 957-959.
- [26] R.G. Novikov, I.A. Taimanov, S.P. Tsarev, Two-dimensional von Neumann-Wigner potentials with a multiple positive eigenvalue, *Functional Analysis and Its Applications*, **48**(4) (2014), 295-297.

- [27] I.A. Taimanov, Blowing up solutions of the modified Novikov-Veselov equation and minimal surfaces, *Theoretical and Mathematical Physics*, **182**(2) (2015), 173-181.
- [28] I.A. Taimanov, The Moutard transformation of two-dimensional Dirac operators and Möbius geometry, *Mathematical Notes*, **97**(1) (2015), 124-135.
- [29] I.A. Taimanov, S.P. Tsarev, On the Moutard transformation and its applications to spectral theory and Soliton equations, *Journal of Mathematical Sciences*, **170**(3) (2010), 371-387.
- [30] I.N. Vekua, *Generalized Analytic Functions*, Pergamon Press Ltd. 1962.